1) $f(x) = \ln(1-x)$

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

with $c_0 = f(0) = \ln(1) = 0$

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{-(n-1)!}{(1-0)^n}\frac{1}{n!} = -\frac{1}{n}$$

so $\ln(1-x) = -\sum_{n=1}^{\infty} x^n/n$

Compute $f'(x) = \frac{-1}{1-x}$

$$f''(x) = \frac{-1}{(1-x)^2}$$

$$f^{(3)}(x) = \frac{-2}{(1-x)^3}$$

$$f^{(4)}(x) = \frac{-6}{(1-x)^4}$$

In general, $f^{(n)}(x) = \frac{-(n-1)!}{(1-x)^n}$

2) $f(x) = \cos(x)$

Compute $f'(x) = -\sin(x)$

$$f''(x) = -\cos(x)$$

$$f^{(3)}(x) = \sin(x)$$

$$f^{(4)}(x) = \cos(x)$$

$$\vdots$$

so $c_n = \begin{cases} 0 & n \text{ odd} \\ \frac{(-1)^{n/2}}{n!} & n \text{ even} \end{cases}$

Then $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

3) $f(x) = \cos(\sqrt{x}) = \sum_{n=0}^{\infty} \frac{(-1)^n (\sqrt{x})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}$

4) $f(x) = \frac{1}{1+2x^3} = \frac{1}{1-(-2x^3)} = \sum_{n=0}^{\infty} (-2x^3)^n = \sum_{n=0}^{\infty} (-1)^n 2^n x^{3n}$

5) $f(x) = \frac{1}{3+x} = \frac{1}{3}\left(\frac{1}{1+\frac{1}{3}x}\right) = \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{1}{3}x\right)^n = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n \frac{x^n}{3^n}$

compare to $= \frac{1}{1-(-2-x)} = \sum_{n=0}^{\infty}(-2-x)^n = \underbrace{\sum_{n=0}^{\infty}(-1)^n (x+2)^n}_{\text{not at } x=0}$

6) a. $f(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

b. $\arctan(x) = \int \frac{1}{1+x^2} dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx$

$$= \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \right) + c$$

with $c = \arctan(0) = 0$

so $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$